

Stability Tests for Second Order Linear and Nonlinear Delayed Models

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Abstract

For the nonlinear second order Lienard-type equations with time-varying delays

$$\ddot{x}(t) + \sum_{k=1}^m f_k(t, x(t), \dot{x}(g_k(t))) + \sum_{k=1}^l s_k(t, x(h_k(t))) = 0,$$

global asymptotic stability conditions are obtained. The results are based on the new sufficient stability conditions for relevant linear equations and are applied to derive explicit stability conditions for the nonlinear Kaldor-Kalecki business cycle model. We also explore multistability of the sunflower non-autonomous equation and its modifications.

AMS Subject Classification: 34K20, 92D25, 34K45, 34K12, 34K25

Keywords: Second order delay differential equations, global asymptotic stability, boundedness of solutions, Lienard-type nonautonomous linear and nonlinear delay differential equations, sunflower equation, the Kaldor-Kalecki business cycle model, variable delays

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[†]Research supported by a grant from VIU

1 Introduction

The second order delay differential equation

$$\ddot{x} + f(t, x(t), \dot{x}(t - \tau)) + g(t, x(t), x(t - \tau)) = 0 \quad (1.1)$$

has a more than 65-year history of study, and was used to examine aftereffects in mechanics, physics, biology, medicine and economics (see, for example, [18]). Recently, these models have been used to mimic regenerative vibrations in a milling process, a balancing motion and chatter vibrations. For example, a one degree of freedom milling equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = -\alpha [x(t) - x(t - \tau(t))] \quad (1.2)$$

was introduced in [35]. The milling model with several delays

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + \sum_{k=1}^p \alpha_k [x(t) - x(t - \tau_k)]^k = 0$$

was recently studied, mostly numerically, in [12, 19, 20]. The following milling models with variable parameters were derived and examined in [18, 27, 28, 35, 36, 37]:

$$\ddot{x}(t) + a\dot{x}(t) + b(t)x(t) = c(t)x(t - \tau(t)), \quad (1.3)$$

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + \sum_{k=1}^p \alpha_k(t)[x(t) - x(t - \tau_k(t))]^k = 0.$$

In economics, the well-known Kaldor-Kalecki business cycle model expressed as the delayed system of two nonlinear equations [15], in some cases can be reduced to the second order equation (see, for example, [29])

$$\ddot{x}(t) + [\alpha - \beta p'(x(t))]\dot{x}(t) + \gamma[p(x(t)) - \eta x(t)] + \delta p(x(t - \tau)) = 0. \quad (1.4)$$

Here $p(x)$ is a frequently used in mathematical economics sigmoid function [15], e.g. $p(x) = \frac{A}{1+e^{-bx}} - \frac{A}{2}$, and all coefficients are nonnegative constants.

Different techniques were applied to study second-order delay equations in [5, 6, 10, 16, 21, 22, 25] and [30]–[34]. Characteristic quasipolynomials were broadly used for local stability analysis of autonomous models, (see, for example, [18]). The fixed point technique for second order differential and functional equations was pioneered by T. A. Burton [7, 8]. In the paper [9] explicit and easily-verifiable tests were obtained for the autonomous model

$$\ddot{x}(t) = p_1\dot{x}(t) + p_2\dot{x}(t - \tau) + q_1x(t) + q_2x(t - \tau). \quad (1.5)$$

Theorem 1.1. [9] *Assume that at least one of the following conditions holds: 1) $p_1p_2 > 0, q_1 > 0, q_2 > 0$ or 2) $p_1 > 0, p_2 > 0, q_1 > 0, q_2 < 0$. Then equation (1.5) is unstable.*

Theorem 1.2. [9] Assume $p_1 = p_2 = 0$, $q_2 > 0$ and denote $B = \tau^2 q_1$, $D = \tau^2 q_2$. Equation (1.5) is asymptotically stable if and only if $q_1 < 0$ and there exists $k \in \mathbb{N}$ such that

$$2k\pi < \sqrt{-B} < (2k+1)\pi, \quad D < \min \{ -(2k)^2 \pi^2 - B, (2k+1)^2 \pi^2 + B \}.$$

Example 1.3. The second-order delay equation

$$\ddot{x}(t) = -49x(t) + 7x(t-1) \quad (1.6)$$

is asymptotically stable by Theorem 1.2. Based on the algorithmic tests presented in [9], the equation

$$\ddot{x}(t) = 0.6\dot{x}(t) + 0.3\dot{x}(t-1) - 2x(t) + x(t-1) \quad (1.7)$$

is asymptotically stable. It is interesting to note that equations (1.6) and (1.7) without delays are unstable. This illustrates a very interesting feature of second-order delay differential equations, i.e. delays may improve asymptotic properties of a given equation, whereas delays in first-order linear equations have mostly destabilizing effects or do not change stability of the model.

Several stability tests for non-autonomous linear models with variable delays

$$\ddot{x}(t) + a(t)\dot{x}(g(t)) + b(t)x(h(t)) = 0, \quad (1.8)$$

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + a_1(t)\dot{x}(g(t)) + b_1(t)x(h(t)) = 0, \quad (1.9)$$

were obtained in our recent paper [3], under the assumptions: a , a_1 , b and b_1 are Lebesgue measurable and essentially bounded functions on $[0, \infty)$; $a(t) \geq a_0 > 0$, $b(t) \geq b_0 > 0$, $0 \leq t - h(t) \leq \tau$, $0 \leq t - g(t) \leq \delta$, $a^2(t) \geq 4b(t)$, $\int_{g(t)}^t a(s)ds < 1/e$. Below $\|\cdot\|$ is the norm in the space $\mathbf{L}_\infty[t_0, \infty)$.

Theorem 1.4. [3, Theorem 5.1] If for some $t_0 \geq 0$

$$\delta \left\| \frac{a}{b} \right\| \left(\|a\| \left\| \frac{b}{a} \right\| + \|b\| \right) + \tau \left\| \frac{b}{a} \right\| < 1,$$

then equation (1.8) is exponentially stable.

Theorem 1.5. [3, Theorem 5.3] Suppose for some $t_0 \geq 0$

$$\left\| \frac{a_1}{a} \right\| < 1, \quad \left\| \frac{a_1}{b} \right\| \frac{\left\| \frac{b}{a} \right\| + \left\| \frac{b_1}{a} \right\|}{1 - \left\| \frac{a_1}{a} \right\|} + \left\| \frac{b_1}{b} \right\| < 1,$$

then equation (1.9) is exponentially stable.

In the present paper, a specially designed substitution transforms linear second order equations into a system, with a further application of the M-matrix method. This and the linearization techniques are used to devise new global stability tests for nonlinear non-autonomous models. These results are explicit, easily verifiable and can be applied to a

general class of second order non-autonomous equations. Some of the theorems of the present paper complement our earlier results [2, 3], as well as the tests obtained in recent papers [9, 10, 16].

The paper is organized as follows. Section 2 contains stability results for linear second order non-autonomous equations with several delays. To illustrate efficiency of the results obtained each stability test is accompanied by numerical examples. In Section 3 the tests for linear models are applied to nonlinear Lienard-type equations of the second order. Applications incorporate a global stability test for the non-autonomous business cycle model. Section 4 includes the study of bounds and multistability properties for the sunflower model and its generalizations. In particular, sufficient conditions for convergence to one of an infinite number of equilibrium points are presented, and existence of unbounded linearly growing solutions is illustrated. Final remarks are presented in Section 5.

2 Stability tests for linear Lienard equations

The technique in this section involves parlaying a second order equation into two first order equations. Consider a linear equation of the second order

$$\ddot{x}(t) + \sum_{k=1}^m a_k(t) \dot{x}(h_k(t)) + \sum_{k=1}^m b_k(t) \int_{g_k(t)}^t \dot{x}(s) ds + \sum_{k=1}^m c_k(t) x(r_k(t)) = 0. \quad (2.1)$$

Together with equation (2.1), for any $t_0 \geq 0$ we consider the initial condition

$$x(t) = \varphi(t), \quad \dot{x}(t) = \psi(t), \quad t \leq t_0. \quad (2.2)$$

Henceforth, we assume that the following assumptions are satisfied:

- (a1) $a_i, b_i, c_i, i = 1, \dots, m$ are Lebesgue measurable and essentially bounded on $[0, \infty)$;
- (a2) h_i, g_i, r_i are Lebesgue measurable functions, $h_i(t) \leq t, g_i(t) \leq t, r_i(t) \leq t$,
 $\lim_{t \rightarrow \infty} h_i(t) = \infty, \lim_{t \rightarrow \infty} g_i(t) = \infty, \lim_{t \rightarrow \infty} r_i(t) = \infty, i, j = 1, \dots, m$;
- (a3) φ and ψ are Borel measurable bounded functions.

Definition 2.1. *A function $x : \mathbb{R} \rightarrow \mathbb{R}$ with locally absolutely continuous on $[t_0, \infty)$ derivative \dot{x} is called a **solution** of problem (2.1), (2.2) if it satisfies equation (2.1) for almost every $t \in [t_0, \infty)$ and equalities (2.2) for $t \leq t_0$.*

We quote a useful lemma that will play a major role in the proofs.

Lemma 2.2. [4] *Consider the system*

$$\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^m \sum_{k=1}^{l_{ij}} b_{ij}^k(t)x_j(h_{ij}^k(t)), \quad i = 1, \dots, m, \quad (2.3)$$

where $a_i(t) \geq \alpha_i > 0$, $|b_{ij}^k(t)| \leq L_{ij}^k$, $t - h_{ij}^k(t) \leq \sigma_{ij}^k$. If the matrix $B = (b_{ij})_{i,j=1}^m$, with $b_{ii} = 1 - \left(\sum_{k=1}^{l_{ii}} L_{ii}^k \right) / \alpha_i$, $b_{ij} = - \left(\sum_{k=1}^{l_{ij}} L_{ij}^k \right) / \alpha_i$, $i \neq j$, is an M -matrix, then system (2.3) is exponentially stable.

We recall that a matrix $B = (b_{ij})_{i,j=1}^m$ is a (nonsingular) M -matrix if $b_{ij} \leq 0$, $i \neq j$ and one of the following equivalent conditions holds: either there exists a positive inverse matrix $B^{-1} > 0$ or all the principal minors of the matrix B are positive.

Further proofs will also require the following lemma.

Lemma 2.3. *Consider the system*

$$\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^m \sum_{k=1}^{l_{ij}} \left(c_{ij}^k(t)x_j(g_{ij}^k(t)) + d_{ij}^k(t) \int_{h_{ij}^k(t)}^t x_j(s)ds \right), \quad i = 1, \dots, m, \quad (2.4)$$

where $a_i(t) \geq \alpha_i > 0$, $|d_{ij}^k(t)| \leq L_{ij}^k$, $|c_{ij}^k(t)| \leq C_{ij}^k$, $t - h_{ij}^k(t) \leq \sigma_{ij}^k$, $t - g_{ij}^k(t) \leq \tau$. If the matrix $B = (b_{ij})_{i,j=1}^m$, with $b_{ii} = 1 - \sum_{k=1}^{l_{ii}} (L_{ii}^k \sigma_{ii}^k + C_{ii}^k) / \alpha_i$, $b_{ij} = - \sum_{k=1}^{l_{ij}} (L_{ij}^k \sigma_{ij}^k + C_{ij}^k) / \alpha_i$, $i \neq j$, is an M -matrix, then system (2.4) is exponentially stable.

Proof. Let $x(t)$ be a solution of (2.4). Since $x_j(t)$ are continuous then for any i, j, k and t there exists $p_{ij}^k(t) \in (h_{ij}^k(t), t)$ such that $x_j(p_{ij}^k(t))(t - h_{ij}^k(t)) = \int_{h_{ij}^k(t)}^t x_j(s)ds$.

Thus x_j are solutions of system (2.3) with $b_{ij}^k(t)x_j(h_{ij}^k(t))$ being replaced by $c_{ij}^k x_j(g_{ij}^k(t)) + d_{ij}^k(t)(t - h_{ij}^k(t))x_j(p_{ij}^k(t))$. We have $|c_{ij}^k(t)| \leq C_{ij}^k$, $|d_{ij}^k(t)(t - h_{ij}^k(t))| \leq L_{ij}^k \sigma_{ij}^k$, $i \neq j$. The application of Lemma 2.2 validates the proof. \square

To examine the equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + \sum_{k=1}^m c_k(t)x(h_k(t)) = 0 \quad (2.5)$$

we assume

$$0 < a \leq a(t) \leq A, \quad 0 < b \leq b(t) \leq B, \quad |c_k(t)| \leq C_k, \quad t - h_k(t) \leq \tau.$$

Theorem 2.4. *Suppose at least one of the following conditions holds:*

- 1) $B \leq \frac{a^2}{4}$, $\sum_{k=1}^m C_k < b - \frac{a}{2}(A - a)$,
- 2) $b \geq \frac{a}{2} \left(A - \frac{a}{2} \right)$, $\sum_{k=1}^m C_k < \frac{a^2}{2} - B$.

Then equation (2.5) is exponentially stable.

Proof. Substituting $\dot{x} = -\frac{a}{2}x + y$, $\ddot{x} = -\frac{a}{2}\dot{x} + \dot{y}$ into equation (2.5), we arrive at

$$\begin{aligned}\dot{x} &= -\frac{a}{2}x + y \\ \dot{y} &= \left[\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - b(t) \right] x(t) - \sum_{k=1}^m c_k(t)x(h_k(t)) - \left(a(t) - \frac{a}{2} \right) y(t).\end{aligned}\tag{2.6}$$

Condition 1) yields $\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - b(t) \geq \frac{a^2}{4} - B > 0$, $\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - b(t) \leq \frac{a}{2} \left(A - \frac{a}{2} \right) - b$. Hence the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \\ -\frac{2}{a} \left(\frac{a}{2} \left(A - \frac{a}{2} \right) - b + \sum_{k=1}^m C_k \right) & 1 \end{pmatrix}$$

is an M-matrix. By Lemma 2.2 equation (2.5) is exponentially stable.

If condition 2) holds then $b(t) - \frac{a}{2} \left(a(t) - \frac{a}{2} \right) \geq b - \frac{a}{2} \left(A - \frac{a}{2} \right) > 0$, $b(t) - \frac{a}{2} \left(a(t) - \frac{a}{2} \right) \leq B - \frac{a}{2} \left(a - \frac{a}{2} \right) = B - a^2/4$, and the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \\ -\frac{2}{a} \left(B - \frac{a^2}{4} + \sum_{k=1}^m C_k \right) & 1 \end{pmatrix}$$

is an M-matrix. By Lemma 2.2 equation (2.5) is exponentially stable. \square

Example 2.5. Consider the delay equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + cx(t-h|\sin t|) = 0\tag{2.7}$$

The following numerical examples illustrate the application of Theorem 2.4:

a) $a = 3, b = 1.1, c = -0.8, h = 2$. Condition 1) of Theorem 2.4 holds, condition 2) does not hold. Equation (2.7) is asymptotically stable.

b) $a = 2, b = 1.1, c = -0.8, h = 2$. Condition 2) of Theorem 2.4 holds, condition 1) does not hold. Equation (2.7) is asymptotically stable.

c) $a = 0.1, b = 1.5, c = -1.45, h = 2$. Conditions of Theorem 2.4 do not hold, and equation (2.7) is unstable. Hence, in general, the conditions $a(t) \geq a_0 > 0$, $b(t) \geq b_0 > 0$, $m = 1$, $|c(t)| < b(t)$ are not sufficient for stability of equation (2.5); however, for the first order differential equation

$$\dot{x}(t) + b(t)x(t) + c(t)x(h(t)) = 0$$

there are sufficient exponential stability conditions for any $h(t)$ satisfying $t - \tau \leq h(t) \leq t$.

Consider the equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + \sum_{k=1}^m c_k(t)\dot{x}(h_k(t)) = 0,\tag{2.8}$$

where

$$0 < a \leq a(t) \leq A, \quad 0 < b \leq b(t) \leq B, \quad |c_k(t)| \leq C_k, \quad t - h_k(t) \leq \tau.$$

Theorem 2.6. *Suppose that at least one of the following conditions holds:*

- 1) $B \leq \frac{a^2}{4}, \quad \sum_{k=1}^m C_k < \frac{2b - a(A - a)}{2a},$
- 2) $b \geq \frac{a}{2} \left(A - \frac{a}{2} \right), \quad \sum_{k=1}^m C_k < \frac{a^2 - 2B}{2a}.$

Then equation (2.8) is exponentially stable.

Proof. The substitution $\dot{x} = -\frac{a}{2}x + y, \ddot{x} = -\frac{a}{2}\dot{x} + \dot{y}$ into equation (2.8) yields

$$\begin{aligned} \dot{x} &= -\frac{a}{2}x + y \\ \dot{y} &= \left[\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - b(t) \right] x(t) + \frac{a}{2} \sum_{k=1}^m c_k(t) x(h_k(t)) \\ &\quad - \sum_{k=1}^m c_k(t) y(h_k(t)) - \left(a(t) - \frac{a}{2} \right) y(t). \end{aligned} \tag{2.9}$$

If condition 1) holds, we have $\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - b(t) \geq \frac{a^2}{4} - B > 0,$

$\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - b(t) \leq \frac{a}{2} \left(A - \frac{a}{2} \right) - b.$ Hence the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \sum_{k=1}^m C_k \\ -\frac{2}{a} \left[\frac{a}{2} \left(A - \frac{a}{2} \right) - b + \frac{a}{2} \sum_{k=1}^m C_k \right] & 1 - \frac{2}{a} \sum_{k=1}^m C_k \end{pmatrix}$$

is an M-matrix. By Lemma 2.2 equation (2.8) is exponentially stable.

If the inequalities in 2) hold then $b(t) - \frac{a}{2} \left(a(t) - \frac{a}{2} \right) \geq b - \frac{a}{2} \left(A - \frac{a}{2} \right) > 0,$
 $b(t) - \frac{a}{2} \left(a(t) - \frac{a}{2} \right) \leq B - \frac{a}{2} \left(a - \frac{a}{2} \right) = B - a^2/4.$ Thus the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \sum_{k=1}^m C_k \\ -\frac{2}{a} \left[B - \frac{a^2}{4} + \frac{a}{2} \sum_{k=1}^m C_k \right] & 1 - \frac{2}{a} \sum_{k=1}^m C_k \end{pmatrix}$$

is an M-matrix. By Lemma 2.2 equation (2.8) is exponentially stable. □

Example 2.7. *Consider the equation*

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + c\dot{x}(t - h|\sin t|) = 0 \tag{2.10}$$

To illustrate Theorem 2.6, we examined:

a) $a = 2.1$, $b = 1$, $c = -0.4$, $h = 2$. Condition 1) of Theorem 2.6 holds, condition 2) does not hold. Equation (2.10) is asymptotically stable.

b) $a = 4$, $b = 5$, $c = -0.7$, $h = 2$. Condition 2) of Theorem 2.6 holds, condition 1) does not hold. Equation (2.10) is asymptotically stable.

c) $a = 1$, $b = 1.5$, $c = -0.8$, $h = 2$. Conditions of the Theorem 2.6 do not hold, and equation (2.10) is unstable. Hence, in general, the conditions $a(t) \geq a_0 > 0$, $b(t) \geq b_0 > 0$, $m = 1$, $|c(t)| < a(t)$ are not sufficient for stability of equation (2.8).

Consider the equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + \sum_{k=1}^m b_k(t)x(h_k(t)) = 0, \quad (2.11)$$

where $0 < a \leq a(t) \leq A$, $0 < b_k \leq b_k(t) \leq B_k$, $t - h_k(t) \leq \tau_k$.

Theorem 2.8. Suppose at least one of the following conditions holds:

- 1) $\sum_{k=1}^m B_k \leq \frac{a^2}{4}$, $\frac{a}{2}(A - a) < \sum_{k=1}^m b_k - a \sum_{k=1}^m B_k \tau_k$,
- 2) $\sum_{k=1}^m b_k \geq \frac{a}{2} \left(A - \frac{a}{2} \right)$, $\sum_{k=1}^m B_k (1 + a \tau_k) < \frac{a^2}{2}$.

Then equation (2.11) is exponentially stable.

Proof. With the substitution $\dot{x} = -\frac{a}{2}x + y$, $\ddot{x} = -\frac{a}{2}\dot{x} + \dot{y}$ into equation (2.11), we arrive at

$$\begin{aligned} \dot{x} &= -\frac{a}{2}x + y \\ \dot{y} &= \left[\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - \sum_{k=1}^m b_k(t) \right] x(t) \\ &\quad + \sum_{k=1}^m b_k(t) \int_{h_k(t)}^t \left[-\frac{a}{2}x(s) + y(s) \right] ds - \left(a(t) - \frac{a}{2} \right) y(t). \end{aligned} \quad (2.12)$$

If condition 1) holds, we have $\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - \sum_{k=1}^m b_k(t) \geq \frac{a^2}{4} - \sum_{k=1}^m B_k > 0$,

$\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - \sum_{k=1}^m b_k(t) \leq \frac{a}{2} \left(A - \frac{a}{2} \right) - \sum_{k=1}^m b_k$. Hence the off-diagonal entries of the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \sum_{k=1}^m B_k \tau_k \\ -\frac{2}{a} \left[\frac{a}{2} \left(A - \frac{a}{2} \right) - \sum_{k=1}^m b_k + \frac{a}{2} \sum_{k=1}^m B_k \tau_k \right] & 1 - \frac{2}{a} \sum_{k=1}^m B_k \tau_k \end{pmatrix}$$

are negative, and the inequalities in 1) yield that it is an M-matrix. By Lemma 2.3 equation

(2.11) is exponentially stable. Assumption 2) implies $\sum_{k=1}^m b_k(t) - \frac{a}{2} \left(a(t) - \frac{a}{2} \right) \geq \sum_{k=1}^m b_k -$

$\frac{a}{2} \left(A - \frac{a}{2} \right) > 0$, $\sum_{k=1}^m b_k(t) - \frac{a}{2} \left(a(t) - \frac{a}{2} \right) \leq \sum_{k=1}^m B_k - \frac{a}{2} \left(a - \frac{a}{2} \right) = \sum_{k=1}^m B_k - a^2/4$, therefore the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \sum_{k=1}^m B_k \tau_k \\ -\frac{2}{a} \left[\sum_{k=1}^m B_k - \frac{a^2}{4} + \frac{a}{2} \sum_{k=1}^m B_k \tau_k \right] & 1 - \frac{2}{a} \sum_{k=1}^m B_k \tau_k \end{pmatrix}$$

is an M-matrix. By Lemma 2.3 equation (2.11) is exponentially stable. \square

Corollary 2.9. Suppose $a(t) \equiv a > 0$, $b_k(t) \equiv b_k > 0$, and at least one of the following conditions holds:

- 1) $\sum_{k=1}^m b_k \leq \frac{a^2}{4}$, $\sum_{k=1}^m b_k(1 - a\tau_k) > 0$,
- 2) $\sum_{k=1}^m b_k \geq \frac{a^2}{4}$, $\sum_{k=1}^m b_k(1 + a\tau_k) < \frac{a^2}{2}$.

Then equation (2.11) is exponentially stable.

Example 2.10. Consider the equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t - h|\sin t|) = 0. \quad (2.13)$$

To illustrate Theorem 2.8, we consider numerical examples:

- a) $a = 2$, $b = 0.9$, $h = 0.4$. Condition 1) of Theorem 2.8 holds, condition 2) does not hold. Equation (2.13) is asymptotically stable.
- b) $a = 2$, $b = 1.1$, $h = 0.4$. Condition 2) of Theorem 2.8 holds, condition 1) does not hold. Equation (2.13) is asymptotically stable.
- c) $a = 1$, $b = 1.1$, $h = 2.5$. Conditions of Theorem 2.8 do not hold. Equation (2.12) is unstable.

Consider the equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = \sum_{k=1}^m c_k(t) [x(t) - x(h_k(t))]. \quad (2.14)$$

where $0 < a \leq a(t) \leq A$, $0 < b \leq b_k(t) \leq B$, $|c_k(t)| \leq C_k$, $t - h_k(t) \leq \tau_k$.

Theorem 2.11. Suppose at least one of the following conditions holds:

- 1) $B \leq \frac{a^2}{4}$, $\sum_{k=1}^m C_k \tau_k < \frac{2b - a(A - a)}{2a}$,
- 2) $b \geq \frac{a}{2} \left(A - \frac{a}{2} \right)$, $\sum_{k=1}^m C_k \tau_k < \frac{a^2 - 2B}{2a}$.

Then equation (2.14) is exponentially stable.

Proof. After rewriting equation (2.14) in the form

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = \sum_{k=1}^m c_k(t) \int_{h_k(t)}^t \dot{x}(s)ds,$$

we apply the same argument as in the proof of Theorem 2.6. \square

Theorem 2.4 gives delay-independent stability conditions for equation (2.5). The following statement contains delay-dependent stability conditions for this equation.

Theorem 2.12. *Assume that*

$$0 < a \leq a(t) \leq A, 0 < b \leq b(t) + \sum_{k=1}^m c_k(t) \leq B, |c_k(t)| \leq C_k, t - h_k(t) \leq \tau_k$$

and at least one of the conditions of Theorem 2.11 holds. Then equation (2.5) is exponentially stable.

Proof. Rewrite equation (2.5) in the form

$$\ddot{x}(t) + a(t)\dot{x}(t) + \left(b(t) + \sum_{k=1}^m c_k(t) \right) x(t) = \sum_{k=1}^m c_k(t) \int_{h_k(t)}^t \dot{x}(s)ds.$$

The end of the proof is a straightforward imitation of the proof of Theorem 2.6. \square

3 Stability tests for nonlinear Lienard equations

In this section we examine several nonlinear delay differential equations of the second order which have the following general form

$$\ddot{x}(t) + \sum_{k=1}^m f_k(t, x(p_k(t)), \dot{x}(g_k(t))) + \sum_{k=1}^l s_k(t, x(h_k(t))) = 0, \quad (3.1)$$

with the following initial function

$$x(t) = \varphi(t), \dot{x}(t) = \psi(t), t \leq t_0, t_0 \geq 0 \quad (3.2)$$

where $f_k(t, u_1, u_2), k = 1, \dots, m, s_k(t, u)$, are Caratheodory functions which are measurable in t and continuous in all the other arguments, condition (a2) holds for delay functions p_k, g_k, h_k ; φ and ψ are Borel measurable bounded functions.

The definition of the solution of the initial value problem (3.1)-(3.2) is the same as for problem (2.1), (2.2). We will assume that the initial value problem has a unique global solution on $[t_0, \infty)$ for all nonlinear equations considered in this section.

Theorem 3.1. *Consider the equation*

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t)) + s(t, x(t)) + \sum_{k=1}^m s_k(t, x(t), x(h_k(t))) = 0, \quad (3.3)$$

where

$$f(t, v, 0) = 0, s(t, 0) = 0, s_k(t, v, 0) = 0, 0 < a_0 \leq \frac{f(t, v, u)}{u} \leq A, \\ 0 < b_0 \leq \frac{s(t, u)}{u} \leq B, \left| \frac{s_k(t, v, u)}{u} \right| \leq C_k, u \neq 0, t - h_k(t) \leq \tau.$$

If at least one of the following conditions holds:

- 1) $B \leq \frac{a_0^2}{4}, \sum_{k=1}^m C_k < b_0 - \frac{a_0}{2}(A - a_0),$
- 2) $b_0 \geq \frac{a_0}{2} \left(A - \frac{a_0}{2} \right), \sum_{k=1}^m C_k < \frac{a_0^2}{2} - B,$

then zero is a global attractor for all solutions of problem (3.3), (3.2).

Proof. First, by the previous theorem there exists a global solution x of problem (3.3), (3.2). Suppose x is a fixed solution of problem (3.3), (3.2). Rewrite equation (3.3) in the form

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + \sum_{k=1}^m c_k(t)x(h_k(t)) = 0,$$

$$\text{where } a(t) = \begin{cases} \frac{f(t, x(t), \dot{x}(t))}{\dot{x}(t)}, & \dot{x}(t) \neq 0, \\ a_0, & \dot{x}(t) = 0, \end{cases} \quad b(t) = \begin{cases} \frac{s(t, x(t))}{x(t)}, & x(t) \neq 0, \\ b_0, & x(t) = 0, \end{cases} \\ c_k(t) = \begin{cases} \frac{s_k(t, x(t), x(h_k(t)))}{x(h_k(t))}, & x(h_k(t)) \neq 0, \\ 0, & x(h_k(t)) = 0. \end{cases}$$

Hence the function x is a solution of the linear equation

$$\ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) + \sum_{k=1}^m c_k(t)y(h_k(t)) = 0, \quad (3.4)$$

which is exponentially stable by Theorem 2.4. Thus for any solution y of equation (3.4) we have $\lim_{t \rightarrow \infty} y(t) = 0$. Since x is a solution of (3.4), we have $\lim_{t \rightarrow \infty} x(t) = 0$. \square

The previous proof is readily adapted to the proof of the following theorems.

Theorem 3.2. *Consider the equation*

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t)) + s(t, x(t)) + \sum_{k=1}^m s_k(t, x(t), \dot{x}(h_k(t))) = 0, \quad (3.5)$$

where

$$f(t, v, 0) = 0, \quad s(t, 0) = 0, \quad s_k(t, v, 0) = 0, \quad 0 < a_0 \leq \frac{f(t, v, u)}{u} \leq A,$$

$$0 < b_0 \leq \frac{s(t, u)}{u} \leq B, \quad \left| \frac{s_k(t, v, u)}{u} \right| \leq C_k, \quad u \neq 0, \quad t - h_k(t) \leq \tau.$$

Suppose at least one of the following conditions holds:

$$1) \quad B \leq \frac{a_0^2}{4}, \quad \sum_{k=1}^m C_k < \frac{2b_0 - a_0(A - a_0)}{2a_0},$$

$$2) \quad b_0 \geq \frac{a_0}{2} \left(A - \frac{a_0}{2} \right), \quad \sum_{k=1}^m C_k < \frac{a_0^2 - 2B}{2a_0}.$$

Then zero is a global attractor for all solutions of problem (3.5), (3.2).

Theorem 3.3. Consider the equation

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t)) + \sum_{k=1}^m s_k(t, x(h_k(t)), \dot{x}(t)) = 0, \quad (3.6)$$

where

$$f(t, v, 0) = 0, \quad s_k(t, 0, u) = 0, \quad 0 < a_0 \leq \frac{f(t, v, u)}{u} \leq A,$$

$$0 < b_k \leq \frac{s_k(t, v, u)}{v} \leq B_k, \quad u \neq 0, \quad t - h_k(t) \leq \tau.$$

Suppose at least one of the following conditions holds:

$$1) \quad \sum_{k=1}^m B_k \leq \frac{a_0^2}{4}, \quad \frac{a_0}{2} (A - a_0) < \sum_{k=1}^m b_k - a_0 \sum_{k=1}^m B_k \tau_k,$$

$$2) \quad \sum_{k=1}^m b_k \geq \frac{a}{2} \left(A - \frac{a_0}{2} \right), \quad \sum_{k=1}^m B_k (1 + a_0 \tau_k) < \frac{a_0^2}{2}.$$

Then zero is a global attractor for all solutions of problem (3.6), (3.2).

Theorem 3.4. Consider the equation

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t)) + s(t, x(t)) = \sum_{k=1}^m c_k(t) (x(t) - x(h_k(t))), \quad (3.7)$$

where

$$f(t, v, 0) = 0, \quad s(t, 0) = 0, \quad 0 < a_0 \leq \frac{f(t, v, u)}{u} \leq A,$$

$$0 < b_0 \leq \frac{s(t, u)}{u} \leq B, \quad |c_k(t)| \leq C_k, \quad u \neq 0, \quad t - h_k(t) \leq \tau_k.$$

Suppose at least one of the following conditions holds:

$$1) \quad B \leq \frac{a_0^2}{4}, \quad \sum_{k=1}^m C_k \tau_k < \frac{2b_0 - a_0(A - a_0)}{2a_0},$$

$$2) b_0 \geq \frac{a_0}{2} \left(A - \frac{a_0}{2} \right), \quad \sum_{k=1}^m C_k \tau_k < \frac{a_0^2 - 2B}{2a_0}.$$

Then zero is a global attractor for all solutions of problem (3.7),(3.2).

Example 3.5. To illustrate Part 2) of Theorem 3.3, consider the equation

$$\ddot{x}(t) + (1.9 + 0.1 \sin x(t)) \dot{x}(t) + (1.1 + 0.1 \cos x(t)) x(t - 0.19 \sin^2 t) = 0. \quad (3.8)$$

We have $m = 1$, $a_0 = 1.8$, $A = 2$, $b_0 = 1$, $B = 1.2$, $\tau = 0.19$; therefore, all conditions of the theorem hold, hence zero is a global attractor for all solutions of equation (3.8).

Motivated by model (1.4), consider a generalized Kaldor-Kalecki model

$$\ddot{x}(t) + [\alpha(t) - \beta(t)p'(x(t))] \dot{x}(t) + s(t, x(t)) = p(x(t)) - p(x(h(t))), \quad (3.9)$$

where α, β are locally essentially bounded functions, s is a Caratheodory function, p is a locally absolutely continuous nondecreasing function,

$$0 < \alpha_0 \leq \alpha(t) \leq \alpha_1, \quad 0 < \beta_0 \leq \beta(t) \leq \beta_1,$$

$$|p'(t)| \leq C, \quad \alpha_0 - \beta_1 C > 0, \quad 0 < b_0 \leq \frac{s(t, u)}{u} \leq B, \quad t - h(t) \leq \tau.$$

Denote $a_0 = \alpha_0 - \beta_1 C$.

Theorem 3.6. Suppose at least one of the following conditions holds:

$$1) B \leq \frac{a_0^2}{4}, \quad C\tau < \frac{2b_0 - a_0(\alpha_1 - a_0)}{2a_0},$$

$$2) b \geq \frac{a_0}{2} \left(\alpha_1 - \frac{a_0}{2} \right), \quad C\tau < \frac{a_0^2 - 2B}{2a_0}.$$

Then zero is a global attractor for all solutions of problem (3.9),(3.2).

Proof. Suppose x is a fixed solution of problem (3.9),(3.2). There exists a function $\xi(t)$ such that $p(x(t)) - p(h(x(t))) = p'(\xi(t))(x(t) - x(h(t)))$. Denote $\alpha(t) - \beta(t)p'(x(t)) = a(t)$, $p'(\xi(t)) = c(t)$. Hence x is a solution of the following equation

$$\ddot{y}(t) + a(t)\dot{y}(t) + s(t, y(t)) = c(t)(y(t) - y(h(t))). \quad (3.10)$$

Since $p'(x) \geq 0$ then $0 < \alpha_0 - \beta_1 C \leq a(t) \leq \alpha_1$. Equation (3.10) has a form (3.7) with $f(t, x(t), \dot{x}(t)) = a(t)\dot{x}(t)$, $m = 1$. All conditions of Theorem 3.4 hold, hence for any solution of (3.10) we have $\lim_{t \rightarrow \infty} y(t) = 0$. Then also $\lim_{t \rightarrow \infty} x(t) = 0$. \square

4 Sunflower model and its modifications

The sunflower equation was introduced in 1967 by Israelson and Johnson in [17] as a model for the geotropic circumnutations of *Helianthus annuus* and studied in [11, 23, 26]. Historically, it was derived from the following first order delay equation

$$\dot{u} + \frac{b}{\tau} e^{a(1-t/\tau)} \int_{-\infty}^{t-\tau} e^{as/\tau} \sin u(s) ds = 0. \quad (4.1)$$

Taking the derivative of (4.1) we arrive at the sunflower equation

$$\ddot{x} + \frac{a}{\tau}\dot{x} + \frac{b}{\tau}\sin x(t - \tau) = 0, \quad (4.2)$$

for which evidently the results of the previous section are not applicable.

Remark 4.1. *It is interesting to note that a non-delayed version of (4.2)*

$$\ddot{x} + a\dot{x} + b\sin x(t) = 0, \quad (4.3)$$

has a long history, (see, for example, [24]). It is easy to prove boundedness of $x(t)$ and $\frac{dx}{dt}$, the existence of chaotic solutions was justified numerically [14]. However, many important questions for delayed model (4.2) are still left unanswered.

Consider a generalization of model (4.1)

$$\frac{du}{dt} + b \int_{-\infty}^{h(t)} K(t, s) \sin u(s) ds = 0, \quad (4.4)$$

with the initial conditions

$$u(t) = \varphi(t), \quad t \leq 0, \quad (4.5)$$

under the following assumptions:

(b1) $h(t) \leq t - \tau$ for some $\tau > 0$;

(b2) $K(\cdot, \cdot)$ is Lebesgue measurable, $K(t, s) \geq 0$, there exists $a > 0$ such that

$$K(t, s) \leq \frac{1}{\tau} \exp \left\{ -\frac{a}{\tau}(t - s - \tau) \right\} \text{ and } \int_0^\infty dt \int_{-\infty}^{h(t)} K(t, s) ds = \infty;$$

(b3) $\varphi : [-\infty, 0] \rightarrow \mathbb{R}$ is a continuous bounded function.

Theorem 4.2. *Suppose that (b1)-(b3) hold, and the characteristic equation*

$$\lambda^2 \tau - a\lambda + be^{\lambda\tau} = 0 \quad (4.6)$$

has a positive root $\lambda_0 > 0$. Then any solution of (4.4)-(4.5) with the initial conditions satisfying either $\varphi(t) \in (2\pi k, 2\pi k + \pi)$, $k \in \mathbb{N}$, or $\varphi(t) \in (2\pi k - \pi, 2\pi k)$, $k \in \mathbb{N}$, together with $|\varphi(t) - 2\pi k| \leq \varphi(0)e^{-\lambda_0 t}$, $t < 0$, tends to $2\pi k$ as $t \rightarrow \infty$.

Moreover, for $\varphi(t) \in (2\pi k, 2\pi k + \pi)$ the solution is monotone decreasing, while for $\varphi(t) \in (2\pi k - \pi, 2\pi k)$ is monotone increasing.

Proof. First assume $\varphi(t) \in (0, \pi)$. Consider the solution of (4.6) on $[0, \tau]$. For $t \in [0, \tau]$

$$\begin{aligned} \frac{du}{dt} &= -b \int_{-\infty}^{h(t)} K(t, s) \sin(u(s)) ds \geq -b \int_{-\infty}^{t-\tau} K(t, s) \varphi(s) ds \\ &\geq -\varphi(0) \frac{b}{\tau} \int_{-\infty}^{t-\tau} \exp \left\{ -\frac{a}{\tau}(t - s - \tau) \right\} e^{-\lambda_0 s} ds \\ &= -\varphi(0) \frac{b}{\tau} \exp \left\{ -\frac{a}{\tau}(t - \tau) \right\} \int_{-\infty}^{t-\tau} \exp \left\{ \left(\frac{a}{\tau} - \lambda_0 \right) s \right\} ds \\ &= -\varphi(0) \frac{b}{a - \lambda_0 \tau} e^{-\lambda_0(t-\tau)} = -\varphi(0) \lambda_0 e^{-\lambda_0 t}, \end{aligned}$$

since $a - \lambda_0\tau = \frac{b}{\lambda_0}e^{-\lambda_0\tau}$ by (4.6).

Since $u'(t) \geq -\varphi(0)\lambda_0 e^{-\lambda_0 t}$, the solution is not below the curve $y = \varphi(0)e^{-\lambda_0 t}$ on $[0, \tau]$, and $x(\tau) \geq \varphi(0)e^{-\lambda_0\tau}$. Now consider the function $x(\tau)e^{-\lambda_0(t-\tau)}$. By the assumption on the initial function

$$u(t) = \varphi(t) \leq \varphi(0)e^{-\lambda_0 t} \leq u(\tau)e^{-\lambda_0(t-\tau)}, \quad t \in [0, \tau]. \quad (4.7)$$

Consider further the initial problem with a shifted initial point $t_0 = \tau$ instead of $t_0 = 0$. We only have to check that $0 < \varphi(t) \leq \varphi(\tau)e^{-\lambda_0(t-\tau)}$, $t < \tau$. However, this inequality is satisfied for $t \in [0, \tau]$ due to (4.7); and by the assumption on the initial function and (4.7) we have

$$0 < \varphi(t) \leq \varphi(0)e^{-\lambda_0 t} \leq u(\tau)e^{-\lambda_0(t-\tau)}, \quad t < \tau.$$

Continuing this process we obtain $u(t) > 0$ for any t . Since u is decreasing for $t \geq 0$, there is $\lim_{t \rightarrow \infty} u(t) = d$. Assuming $d > 0$ we obtain from $\int_0^\infty dt \int_{-\infty}^{h(t)} K(t, s) ds = \infty$ in (b2) that $\lim_{t \rightarrow \infty} u(t) = -\infty$, which is a contradiction. A similar argument proves the case $\varphi(t) \in (-\pi, 0)$. If $\varphi(t) \in (2\pi k - \pi, 2\pi k)$, we apply the same argument to $u - 2\pi k$. \square

Note that sharp conditions when all solutions of characteristic equation (4.6) have positive real parts can be found in [26, Lemma 3.1, p. 470].

Corollary 4.3. *Let*

$$\tau < \frac{a^2}{4b}e^{-a/2} \quad (4.8)$$

and $|\varphi(t) - 2\pi k| \leq \varphi(0)e^{-\lambda_0 t}$, $t < 0$, then any solution of (4.4)-(4.5) with the initial conditions satisfying $\varphi(t) \in (2\pi k, 2\pi k + \pi)$, $k \in \mathbb{N}$, is monotone decreasing and tends to $2\pi k$ as $t \rightarrow \infty$. Any solution with $\varphi(t) \in (2\pi k - \pi, 2\pi k)$, $k \in \mathbb{N}$ tends to $2\pi k$ as $t \rightarrow \infty$.

Proof. Let $f(\lambda) = \tau\lambda^2 - a\lambda + be^{\lambda\tau}$, then $f(0) = b > 0$. Inequality (4.8) implies $f(a/(2\tau)) = -a^2/(4\tau^2) + be^{a/2} < 0$, so equation (4.6) has a positive solution. The application of Theorem 4.2 concludes the proof. \square

The following example illustrates that conditions (b1)-(b3) do not guarantee boundedness of solutions of equation (4.4) with the generalized kernel.

Example 4.4. *Let $a = \frac{1}{3} \ln\left(\frac{4}{\pi}\right)$, $b = 2$, $\tau = \pi$,*

$$K(t, s) = \begin{cases} \frac{1}{4}, & t \in [(2k-1)\pi, (2k+1)\pi], \quad s \in [(2k-3)\pi, (2k-2)\pi], \\ 0, & t \in [(2k-1)\pi, (2k+1)\pi], \quad s \notin [(2k-3)\pi, (2k-2)\pi]. \end{cases}$$

Then obviously $K(t, s) = 0$ for $s > t - \pi = t - \tau$, and also for $t - s > 4\pi$. The exponential estimate has the form

$$0 \leq K(t, s) \leq \frac{1}{\pi} e^{-\frac{1}{3\pi} \ln(4/\pi)(t-s-\pi)} = \frac{1}{\pi} \left(\frac{4}{\pi}\right)^{-(t-s-\pi)/(3\pi)},$$

but as $t-s-\pi \leq 3\pi$ whenever $K(t, s) \neq 0$, the right-hand side is not less than $\frac{1}{\pi} \left(\frac{4}{\pi}\right)^{-1} = \frac{1}{4}$, thus $K(t, s)$ has an exponential estimate as in (b2). Further, $u(t) = t$ is an unbounded solution of (4.4). In fact, let $u(t) = t$, $t \in [-\pi, \pi]$. Then for $t \in [\pi, 3\pi]$ we have $\frac{du}{dt} = -2 \int_{-\pi}^0 \frac{1}{4} \sin(t) dt = 1$, so $u(t) = t$ on $[-\pi, 3\pi]$. Due to the periodicity of the sine function and K , we have $\frac{du}{dt} \equiv 1$. Thus the solution is a linear function $u(t) = t$, and it is unbounded.

In the following theorem we will prove that for nonautonomous case the solution of the sunflower equation is bounded by a linear function.

Consider the non-autonomous sunflower equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)\sin x(h(t)) = 0. \quad (4.9)$$

Theorem 4.5. Suppose $a(t) \geq a_0 > 0$, $|b(t)| \leq b_0$. For any solution $x(t)$ of equation (4.9) we have the estimates

$$|x(t)| \leq |x(t_0)| + \left(|\dot{x}(0)| + \frac{b_0}{a_0} \right) t, \quad |\dot{x}(t)| \leq |\dot{x}(0)| + \frac{b_0}{a_0}.$$

Proof. Denote $\dot{x} = y$, $f(t) = b(t)\sin x(h(t))$, where $|f(t)| \leq b_0$. Then $\dot{y}(t) + a(t)y(t) + f(t) = 0$, hence $y(t) = y(0) + \int_0^t e^{-\int_s^t a(\tau) d\tau} f(s) ds$. Then

$$|\dot{x}(t)| \leq |\dot{x}(0)| + \int_0^t e^{-a_0(t-s)} |f(s)| ds \leq |\dot{x}(0)| + \frac{b_0}{a_0},$$

$$x(t) = x(0) + \int_0^t \dot{x}(s) ds, \quad |x(t)| \leq |x(t_0)| + \left(|\dot{x}(0)| + \frac{b_0}{a_0} \right) t.$$

Local stability conditions for equation (4.9) one can find in the following theorem. □

Theorem 4.6. Suppose $0 < a \leq a(t) \leq A$, $0 < b \leq b(t) \leq B$, $t - h(t) \leq \tau$ and at least one of the following conditions hold:

- 1) $B \leq \frac{a^2}{4}$, $\frac{a}{2}(A - a) < b - aB\tau$,
- 2) $b \geq \frac{a}{2} \left(A - \frac{a}{2} \right)$, $B(1 + a\tau) < \frac{a^2}{2}$.

Then any equilibrium $x(t) = 2k\pi$, $k = 0, \dots$ of equation (4.9) is locally asymptotically stable. Any equilibrium $x(t) = (2k + 1)\pi$, $k = 0, \dots$ is not asymptotically stable.

Proof. For the equilibrium $x(t) = 2k\pi$, the linearization of equation (4.9) has the form

$$\ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(h(t)) = 0,$$

which is asymptotically stable by Theorem 2.8.

For the equilibrium $x(t) = (2k + 1)\pi$, the linearized equation for (4.9) has the form

$$\ddot{y}(t) + a(t)\dot{y}(t) - b(t)y(h(t)) = 0. \quad (4.10)$$

Consider now the ordinary differential equation

$$\ddot{z}(t) + a(t)\dot{z}(t) = 0. \quad (4.11)$$

The fundamental function of equation (4.11) (the solution of initial value problem with $z(0) = 0, z'(0) = 1$) has the form

$$z(t) = \int_0^t e^{-\int_0^s a(\tau) d\tau} ds,$$

which is a positive function for $t > 0$ with a nonnegative derivative. By [1, Theorem 8.3] for the fundamental function $y(t)$ of equation (4.10) we have $y(t) > 0, y'(t) \geq 0$ for $t > 0$. Hence $y(t)$ does not tend to zero, and thus equation (4.10) is not asymptotically stable. \square

5 Concluding Remarks

The technique of reduction of a high-order linear differential equation to a system by the substitution $x^{(k)} = y_{k+1}$ is quite common. However, this substitution does not depend on the parameters of the original equation, and therefore does not offer new insight from a qualitative analysis point of view. Instead, we proposed a substitution which exploits the parameters of the original model. By using that approach, a broad class of the second order non-autonomous linear equations with delays was examined and explicit easily-verifiable sufficient stability conditions were obtained. There is a natural extension of this approach to stability analysis of high-order models. For the nonlinear second order non-autonomous equations with delays we applied the linearization technique and the results obtained for linear models. Our stability tests are applicable to some milling models, e.g. models (1.2) and (1.3); and to a non-autonomous Kaldor–Kalecki business cycle model. Several numerical examples illustrate the application of the stability tests. We suggest that a similar technique can be developed for higher order linear delay equations, with or without non-delay terms. For a nonautonomous version of a classical sunflower model, we verified that the derivative is bounded and thus the solution has a linear bound. Example 4.4 illustrates the existence of an unbounded linearly growing solution for the generalized sunflower equation. We also obtained sufficient conditions under which a solution tends to one of the infinite number of the equilibrium points.

Solution of the following problems will complement the results of the present paper:

1. In all stability conditions obtained, we used lower and upper bounds of the coefficients and the delays. It is interesting to obtain stability conditions in an integral form, for instance, in the assumptions of Theorem 2.8 replace the term $a\tau_k$ by, generally, a smaller term $\int_{h_k(t)}^t a(s) ds$.

2. Apply the technique used in the paper to examine delay differential equations of higher order.
3. Is it possible to generalize Theorem 4.2 to the case when the initial function $\varphi(t) \in (2\pi k - \pi, 2\pi k + \pi)$ and characteristic equation (4.6) has a solution with a positive real part?
4. Establish necessary stability conditions for the equations considered in this paper.
5. For the sunflower equation and its modifications establish set of conditions to guarantee boundedness of all solutions.

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